THE SPECTRUM AND ISOMETRIC EMBEDDINGS OF SURFACES OF REVOLUTION

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For Gus and Sonia

ABSTRACT. An upper bound on the first S^1 invariant eigenvalue of the Laplacian for S^1 invariant metrics on S^2 is used to find obstructions to the existence of isometric embeddings of such metrics in $(\mathbb{R}^3, can)$. As a corollary we prove: If the first four distinct eigenvalues have even multiplicities then the surface of revolution cannot be isometrically embedded in $(\mathbb{R}^3, can)$. This leads to a generalization of a classical result in the theory of surfaces.

1. INTRODUCTION

The problem of isometrically embedding (S^2, g) in $(\mathbb{R}^3, can)$ has a long history which goes back at least as far as 1916. In that year, Weyl, [17], and in the years since, Nirenberg, [15], Heinz, [9], Alexandrov, [2], and Pogorelov, [16], to name a few, proved embedding theorems of various orders of differentiability in case the Gauss curvature is positive. A recent result of Hong and Zuily [11] addresses the case of non-negative curvature. But, of course, not every metric on S^2 admits such an isometric embedding. The reader may refer to Greene, [8], wherein one finds examples of smooth metrics on S^2 for which there is no C^2 isometric embedding in $(\mathbb{R}^3, can)$.

In the presence of examples such as Greene's, one might naturally ask if there exist intrinsic geometric conditions on metrics which obstruct such isometric embeddings. Inasmuch as the above mentioned embedding theorems require, at least, non-negativity of the Gauss curvature, one must look for embedding obstructed metrics among those with some negative curvature. Of course, having some negative curvature is not enough, but one might hope that some stronger condition, associated with the existence of negative curvature at a point, might satisfy our requirements. The purpose of this paper is to provide, in a special case, conditions on the spectrum of the Riemannian manifold which are intrinsic obstructions to the above isometric embedding problem.

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It is no surprise that the spectrum might make an appearance in this subject. There is an extensive literature which associates the spectrum with (more generally) isometric immersions (see, for example, [4] or [14] among others). Much of this work relates the spectrum to the mean curvature, and is associated with the Willmore conjecture. By way of comparison, the embedding problem of this paper is almost trivial, but it is the new and more intrinsic relation between the spectrum and embeddability which we hope the reader will find interesting.

In the case of S^1 invariant metrics on S^2 (i.e. surfaces of revolution), one can prove that, while the first eigenvalue must be bounded above by $8\pi/area$ (Hersch's theorem, [10]), the first S^1 invariant eigenvalue can be arbitrarily large. At the same time, however, there is an upper bound, depending on the metric, for the first S^1 invariant eigenvalue. We will prove that it is this upper bound which, upon exceeding a certain critical value, becomes an obstruction to isometric embedding into $(\mathbb{R}^3, can)$ (Necessarily, the same condition ensures that there is some negative curvature). As a result, if the first S^1 invariant eigenvalue becomes too large then the surface cannot be isometrically embedded in $(\mathbb{R}^3, can)$. (See also Abreu and Freitas [1].)

Another characteristic of the spectrum of a surface of revolution is that the eigenspaces are even dimensional unless the eigenvalue happens to correspond to an S^1 invariant eigenvalue. As a result, one way of increasing the first S^1 invariant eigenvalue is to insist that the multiplicities be even up to a certain point. This leads to a result, proved in Section 5, that even multiplicity for the first 4 distinct eigenvalues is an obstruction to isometric embeddability.

In the last section, we will remark on how these results give a generalization of a well known corollary of the Gauss-Bonnet theorem regarding the existence of points with negative curvature.

The author is indebted to Andrew Hwang for a brief but enlightening conversation about momentum coordinates.

2. Isometric embeddings and Momentum Coordinates

First, we will discuss a formulation of the condition on the metric which ensures that an S^1 invariant metric on S^2 may be isometrically embedded into $(\mathbb{R}^3, can)$. This condition is well known and quite elementary. The reader will find our treatment to be essentially equivalent to that of Besse [3], p. 95-105.

Let (M, g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2. We will assume the metric to be C^∞. Since (M, g) has an effective S^1 isometry group there are exactly two fixed points. We call the fixed points np and sp and let U be the chart $M \setminus \{np, sp\}$. On U the metric has the form $ds \otimes ds + a^2(s) d\theta \otimes d\theta$ where s is the arclength along a geodesic connecting np to sp and $a(s)$ is a function $a : [0, L] \to \mathbb{R}^+$ satisfying $a(0) = a(L) = 0$ and $a'(0) = 1 = -a'(L)$.

It is easy to see that isometric embeddings of such metrics can be parametrized as follows:

$$\left\{\begin{array}{lcl}\psi^1 & = & a(s)\cos\theta \\ \psi^2 & = & a(s)\sin\theta \\ \psi^3 & = & \pm\int^s\sqrt{1-(a')^2(t)}dt\end{array}\right. \tag{2.1}$$

It is evident from this formula that (M,g) can be isometrically C^1 embedded in $(\mathbb{R}^3, can)$ if and only if

$$|a'(s)| \leq 1 \quad \text{for all} \quad s \in [0,L]. \tag{2.2}$$

We will find it convenient to make a change of variables to so called <u>momentum coordinates</u> (See Hwang and Singer [12]). These are given by a diffeomorphism $(s,\theta) \to (x,\theta)$ where $x \equiv \phi : [0,L] \to [-1,1]$ is defined by:

$$x \equiv \phi(s) \equiv \int_c^s a(t)dt. \tag{2.3}$$

If we let $f(x) \equiv (a^2 \circ \phi^{-1})(x)$, then in the new coordinates the metric on the chart U takes the form

$$g = \frac{1}{f(x)} dx \otimes dx + f(x) d\theta \otimes d\theta \tag{2.4}$$

where $(x,\theta) \in (-1,1) \times [0,2\pi)$. In these coordinates the conditions at the endpoints translate to $f(-1) = 0 = f(1)$ and $f'(-1) = 2 = -f'(1)$. In this form, it is easy to see that this metric has area 4π and that its Gauss curvature is given by $K(x) = (-1/2)f''(x)$. It is also worth observing that the function $f(x)$ is the square of the length of the Killing field (infinitesimal isometry) $\partial/\partial\theta$ on the chart U. The canonical (i.e. constant curvature) metric is obtained by taking $f(x) = 1 - x^2$.

Now using (2.2) and the definition for f it is a simple exercise from calculus to prove:

Proposition 2.1. *Let (M,g), with metric g as in (2.4), be diffeomorphic to S^2. (M,g) can be isometrically C^1 embedded in $(\mathbb{R}^3, can)$ if and only if $|f'(x)| \leq 2$ for all $x \in [-1,1]$.* □

We will end this section with the comment that it is also very easy to see that (in our special case) non-negative curvature implies isometric embeddability since $K(x) \geq 0$ implies that $f'(x)$ is a non-increasing function on $[-1,1]$ with maximum 2 and minimum -2. One can find essentially the same comment on p-106 of [3].

3. Some properties of the spectrum

In the interest of presenting a self-contained exposition we will review some of the relevent facts about the spectrum (eigenvalues) of a surface of revolution in this section. The interested reader may consult [5], [6] and [7] for further details.

Let Δ denote the scalar Laplacian on a surface of revolution (M, g), where g is given by (2.4) and let λ be any eigenvalue of $-\Delta$. We will use the symbols E_λ and $\dim E_\lambda$ to denote the eigenspace for λ and it's multiplicity respectively. In this paper the symbol λ_m will always mean the m-th <u>distinct</u> eigenvalue. We adopt the convention $\lambda_0 = 0$. Since S^1 (parametrized here by $0 \leq \theta < 2\pi$) acts on (M, g) by isometries and because $\dim E_{\lambda_m} \leq 2m+1$ (see [7] for the proof), the orthogonal decomposition of E_{λ_m} has the special form

$$E_{\lambda_m} = \bigoplus_{k=-m}^{k=m} e^{ik\theta} W_k$$

in which $W_k(= W_{-k})$ is the "eigenspace" (it might contain only 0) of the ordinary differential operator

$$L_k = -\frac{d}{dx}\left(f(x)\frac{d}{dx}\right) + \frac{k^2}{f(x)}$$

with suitable boundary conditions. It should be observed that $\dim W_k \leq 1$, a value of zero for this dimension occuring when $\lambda_m \notin SpecL_k$.

It is easy to see that $Spec(-\Delta) = \bigcup_{k\in\mathbb{Z}} SpecL_k$, consequently the spectrum of $-\Delta$ can be studied via the spectra $SpecL_k = \{0 < \lambda_k^1 < \lambda_k^2 < \cdots < \lambda_k^j < \cdots\} \forall k \in \mathbb{Z}$. The eigenvalues λ_0^j in the case $k = 0$ above are called the <u>S^1 invariant eigenvalues</u> since their eigenfunctions are invariant under the the S^1 isometry group. If $k \neq 0$ the eigenvalues are called <u>k equivariant</u> or simply <u>of type $k \neq 0$</u>. Each L_k has a Green's operator, $\Gamma_k : (H^0(M))^\perp \to L^2(M)$, whose spectrum is $\{1/\lambda_k^j\}_{j=1}^\infty$, and whose trace is defined by, $tr\Gamma_k \equiv \sum 1/\lambda_k^j$.

Proposition 3.1 (See [5] and [6]). *With the notations as above:*

i:

$$tr\Gamma_k = \begin{cases} \frac{1}{2}\int_{-1}^{1} \frac{1-x^2}{f(x)}dx & \text{if } k = 0 \\ \frac{1}{|k|} & \text{if } k \neq 0 \end{cases}.$$

ii: *If $area(M, g) = 4\pi$ and $\displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_0^j} \leq \frac{\pi^2}{16}$ then there exist points $p \in M$ such that $K(p) < 0$.*

iii: *For all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_k^j = \lambda_{-k}^j$.*

iv: *$\forall k \geq 1$ and $\forall j \geq 0$, $\lambda_{k+j} \leq \lambda_k^{j+1}$; and $\lambda_1 \leq \lambda_0^1$.*

v: $\dim E_{\lambda_m}$ *is odd if and only if* λ_m *is an* S^1 *invariant eigenvalue.* □

Remarks. 1.) One must be careful with the definition of $tr\Gamma_0$ since $\lambda_0 = 0 \in SpecL_0$. To avoid this difficulty we studied the S^1 invariant spectrum of the Laplacian on 1-forms in [6] and then observed that the non-zero eigenvalues are the same for functions and 1-forms.

2.) A slight modification of the proof of Proposition 3.1 ii.) (in [6]) reveals that $\sum_{j=1}^{\infty} \frac{1}{\lambda_0^j} \leq \frac{\pi^2}{16}$ implies that (M, g) cannot be isometrically embedded in $(\mathbb{R}^3, can)$. The reader will find that the results of this paper are an improvement on this idea.

4. A sharp upper bound for the first eigenvalue

In [7] we derived sharp upper bounds for all of the distinct eigenvalues on a surface of revolution diffeomorphic to S^2. These estimates were obtained using the the k-type eigenvalues for $k \neq 0$. In this section we will obtain a sharp bound for λ_1 using the S^1 invariant spectrum. In contrast with the more general result of Hersch [10], the reader will find that this bound exhibits, more explicitly, its dependence on the metric. This fact will play an important rôle in embedding problems.

Proposition 4.1. *Let* (M, g) *be an* S^1 *invariant Riemannian manifold of area* 4π *which is diffeomorphic to* S^2*, with metric* (2.4)*. Let* λ_0^1 *be the first non-zero* S^1 *invariant eigenvalue for this metric, then*

$$\lambda_0^1 \leq \frac{3}{2} \int_{-1}^{1} f(x)dx$$

and equality holds if and only if (M, g) *is isometric to* (S^2, can)*.*

Proof. The minimum principle associated with the first S^1 invariant eigenvalue problem,

$$L_0 u = -\frac{d}{dx}\left(f(x)\frac{du}{dx}\right) = \lambda_0^1 u, \tag{4.1}$$

states that

$$\lambda_0^1 \leq \frac{\int_{-1}^{1} f(x)(\frac{du}{dx})^2 dx}{\int_{-1}^{1} u^2 dx} \tag{4.2}$$

for all S^1 invariant functions $u \in C^\infty(M)$ with $u \perp \ker L_0$. Equality holds if and only if u is an eigenfunction for λ_0^1. Since $\ker L_0$ consists of constant functions and $\int_{-1}^{1} x \cdot 1 dx = 0$, we see that $u(x) = x$ is an admissible solution of (4.2) and therefore $\lambda_0^1 \leq \frac{3}{2} \int_{-1}^{1} f(x)dx$. Equality holds if and only if $u(x) = x$ is the first S^1 invariant eigenfunction. In this case, upon substitution

of $u(x) = x$ into (4.1) we obtain the equivalent equation $-f'(x) = \lambda_0^1 x$. Recalling that $f(x)$ and $f'(x)$ must satisfy certain boundary conditions forces $\lambda_0^1 = 2$ and yields the unique solution $f(x) = 1 - x^2$. In other words, $g = can$. □

Because of Proposition 3.1 iv.), we have the immediate corollary:

Corollary 4.2. *Let (M, g) be an S^1 invariant Riemannian manifold of area 4π which is diffeomorphic to S^2, with metric (2.4). Let λ_1 be the first, non-zero, distinct eigenvalue for this metric, then*

$$\lambda_1 \leq \frac{3}{2} \int_{-1}^{1} f(x)dx$$

and equality holds if and only if (M, g) is isometric to (S^2, can). □

5. Spectral obstructions to isometric embeddings

In [6] we used the trace formula of Proposition 3.1 i.) to show that there exist surfaces of revolution with arbitrarily large first S^1 invariant eigenvalue. This fact, together with Proposition 4.1 of the last section, shows that as λ_0^1 increases so does the integral $\int_{-1}^{1} f(x)dx$. This fact is the key to the results of this section, but first we will prove a lemma which gives lower bounds for our eigenvalues.

Lemma 5.1. *Let $f(x)$ and λ_k^m be defined as above then for all $m \in \mathbb{N}$*

$$\lambda_k^m > \begin{cases} 2m \left[\int_{-1}^{1} \frac{1-x^2}{f(x)} dx\right]^{-1} & \text{if } k = 0 \\ m|k| & \text{if } k \neq 0 \end{cases}.$$

Proof. From Proposition 3.1 i.) $\displaystyle\frac{1}{2}\int_{-1}^{1} \frac{1-x^2}{f(x)} dx = \sum_{j=1}^{\infty} \frac{1}{\lambda_0^j}$ and $\displaystyle\frac{1}{|k|} = \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j}$. Each of the sequences $\left\{\lambda_k^j\right\}_{j=1}^{\infty}$ is positive and strictly increasing so by truncating the above series after m terms and then replacing each term with the smallest one we obtain

$$\frac{1}{2}\int_{-1}^{1} \frac{1-x^2}{f(x)} dx > \frac{m}{\lambda_0^m} \quad \text{and} \quad \frac{1}{|k|} > \frac{m}{\lambda_k^m}.$$

This produces the desired inequalities. □

As was observed in [6], the $k = 0$, $m = 1$ case of this inequality, together with the minimal restrictions on the function f is enough to ensure that there exist surfaces of revolution with arbitrarily large λ_0^1. Because of this, we can be confident that the next two results are non-vacuous.

Proposition 5.2. *Let (M,g) be an S^1 invariant Riemannian manifold of area 4π which is diffeomorphic to S^2 and let λ_0^1 be it's first non-zero S^1 invariant eigenvalue. If $\lambda_0^1 > 3$ then (M,g) cannot be isometrically C^1 embedded in $(\mathbb{R}^3, can)$.*

Proof. By Proposition 4.1, since $\lambda_0^1 > 3$, then $\int_{-1}^1 f(x)dx > 2$. Upon integrating by parts we have $-\int_{-1}^1 xf'(x)dx > 2$ so that

$$2 < \left| -\int_{-1}^1 xf'(x)dx \right| \leq \int_{-1}^1 |x||f'(x)|dx \leq \max_{x\in[-1,1]} |f'(x)|.$$

So there exists $x_0 \in [-1,1]$ with $|f'(x_0)| > 2$, thus, by Proposition 2.1, precluding the possibility of an isometric embedding. $\square$

Since non-embeddable metrics have some negative curvature, we have the immediate corollary:

Corollary 5.3. *Let (M,g) be an S^1 invariant Riemannian manifold of area 4π which is diffeomorphic to S^2, let K be it's Gauss curvature, and let λ_0^1 be it's first S^1 invariant eigenvalue. If $\lambda_0^1 > 3$ then there exists a point $p \in M$ such that $K(p) < 0$.* $\square$

Remarks. Rafe Mazzeo and Steve Zelditch have brought to our attention a recent result of Abreu and Freitas, [1], which is a significant improvement of Proposition 5.2. They prove, with the same hypothesis as Proposition 5.2 and using the notation of this paper, that, for metrics isometrically embedded in $(\mathbb{R}^3, can)$, $\lambda_0^j < \xi_j^2/2$, for all j where ξ_j is a positive zero of a certain Bessel function or its derivative. In particular, $\lambda_0^1 < \xi_1^2/2 \approx 2.89$. We have left Proposition 5.2 in the paper since its proof is so easy, and because the eigenvalue bound contained therein is sufficient for proving the main theorem (Theorem 5.5) below.

As we allow the first S^1 invariant eigenvalue to increase one might suspect that, so to speak, some small eigenvalues with even multiplicity are "left behind". This suggests that we might find an obstruction to embedding if the first few eigenvalues have even multiplicities. We will soon see that even multiplicities for the first four eigenvalues will constitute such an obstruction, but first it would be a good idea to know if metrics with this property exist. This is the subject of:

Theorem 5.4. *There exist metrics on S^2 whose first four distinct non-zero eigenvalues have even multiplicity.*

Proof. To prove this theorem we will find an S^1 invariant metric of area 4π with this property.

By Proposition 3.1 v.), $\dim E_{\lambda_m}$ is even if and only if λ_m is not an S^1 invariant eigenvalue, i.e. if and only if $\lambda_m \neq \lambda_0^j$ for any j. It is now clear that the first four multiplicities are even if and only if $\lambda_4 < \lambda_0^1$, and, by Proposition 3.1, iv.), this will occur if our metric satisfies $\lambda_4^1 < \lambda_0^1$. Using

a variational principle, as in [7], for the operator L_4, we obtain the upper bound:

$$\lambda_4^1 \leq \frac{\int_{-1}^{1}\left[f(x)\left(\frac{du}{dx}\right)^2 + \frac{4^2}{f(x)}u^2\right]dx}{\int_{-1}^{1} u^2 dx}$$

$\forall u \in C^\infty(-1,1)$ such that $u(-1) = u(1) = 0$.

Comparing this upper bound with the lower bound on λ_0^1 provided by Lemma 5.1., the proof of this theorem may now be reduced to finding a function f and a suitable test function u such that

$$\frac{\int_{-1}^{1}\left[f(x)\left(\frac{du}{dx}\right)^2 + \frac{16}{f(x)}u^2\right]dx}{\int_{-1}^{1} u^2 dx} < 2\left[\int_{-1}^{1}\frac{1-x^2}{f(x)}dx\right]^{-1} \tag{5.1}$$

We claim that $f(x) = \dfrac{10(1-x^2)}{1+9x^{36}}$ and $u(x) = \sqrt{1-x^2}$ will satisfy the inequality (5.1).

It is not difficult to see that $2\left[\displaystyle\int_{-1}^{1}\frac{1-x^2}{f(x)}dx\right]^{-1} = \dfrac{185}{23} > 8$ for this choice of $f(x)$. So the right hand side of (5.1) is greater than 8. Calculating the left hand side of (5.1) for this choice of $f(x)$ and $u(x)$ yields:

$$\begin{aligned}\frac{\int_{-1}^{1}\left[f(x)\left(\frac{du}{dx}\right)^2 + \frac{16}{f(x)}u^2\right]dx}{\int_{-1}^{1} u^2 dx} &= \frac{3}{4}\left[10\int_{-1}^{1}\frac{x^2}{1+9x^{36}}dx + \frac{8}{5}\int_{-1}^{1}(1+9x^{36})dx\right] \\ &< \frac{3}{4}\left[10\cdot\frac{2}{3} + \frac{16}{5}\cdot\frac{46}{37}\right] = \frac{1477}{185} < 8,\end{aligned}$$

where the first integral in brackets has been approximated in the obvious way. Since the left hand side is less than 8 and the right hand side is greater than 8, the proof is finished. □

The proof of this theorem is hardly optimal since there are, certainly, many such metrics. We also believe that using a similar technique, one should be able to find metrics whose first m distinct eigenvalues have even multiplicity for arbitrary m, but we will not address these problems here.

Theorem 5.5. *Let (M,g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2 and let λ_m be its m-th distinct eigenvalue. If $\dim E_{\lambda_m}$ is even for $1 \leq m \leq 4$ then (M,g) cannot be isometrically C^1 embedded in $(\mathbb{R}^3, can)$.*

Proof. Without loss of generality, we may assume the area of the metric is 4π. As seen in the proof of Theorem 5.4, the first four eigenvalues have even multiplicity if and only if $\lambda_4 < \lambda_0^1$. This result will then follow from

Proposition 5.2 as long as we can prove that $\lambda_4 > 3$. This is most easily accomplished by contradiction.

Assume $\lambda_4 \leq 3$ so that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leq 3$. Now each λ_i for $1 \leq i \leq 4$ must satisfy $\lambda_i = \lambda_k^l$ for some $k \neq 0$ and $l \geq 1$. However, by Lemma 5.1, $\lambda_k^l > l|k|$ so if $\lambda_i = \lambda_k^l \leq 3$ it must be the case that $l|k| \leq 2$. By Proposition 3.1 iii.) $\lambda_k^l = \lambda_{-k}^l$ so there are only three (possibly) distinct eigenvalues with these properties and their values coincide with λ_1^1, λ_2^1, and λ_1^2. There are, therefore, at most three distinct values for the four distinct eigenvalues λ_i for $1 \leq i \leq 4$, but this contradicts the pigeonhole principle. □

Again there is an immediate corollary:

Corollary 5.6. *Let (M, g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2, let K be it's Gauss curvature, and let λ_m be it's m-th distinct eigenvalue. If $\dim E_{\lambda_m}$ is even for $1 \leq m \leq 4$ then there exists a point $p \in M$ such that $K(p) < 0$,* □

and, contrapositively, a kind of partial converse for Weyl type theorems:

Corollary 5.7. *A surface of revolution which is diffeomorphic to S^2 and isometrically embedded in $(\mathbb{R}^3, can)$ has the property that at least one of its first four non-zero distinct eigenvalues has odd multiplicity.* □

We observe that this is a property which these metrics share with those of constant positive curvature.

6. Remarks on classical surface theory

In this final section we leave behind the question of embeddability and focus our attention on the way in which Corollary 5.6 can be viewed as an extension of one of the corollaries of the Gauss-Bonnet theorem.

Let (M, g) be any compact, orientable, boundaryless surface with metric g. We recall that the Euler characteristic, $\chi(M)$, and curvature K are related by the Gauss Bonnet Theorem:

$$2\pi\chi(M) = \int_M K,$$

so that one has the well known result:

Proposition 6.1. *If $\chi(M) \leq 0$ then there exists a point $p \in M$ such that $K(p) \leq 0$.* □

Via the Hodge-DeRham isomorphism, one can restate the Gauss-Bonnet theorm as follows:

Let $\lambda_{q,j}$ be the j-th distinct eigenvalue of the Laplacian acting on q-forms and $E_{\lambda_{q,j}}$ its "eigenspace" (this vectorspace may consist of the zero vector only). Then

$$\frac{1}{2\pi}\int_M K = 2 - \dim E_{\lambda_{1,0}}. \tag{6.1}$$

Of course $\dim E_{\lambda_{1,0}}$ is simply twice the genus of the surface since $\lambda_{1,0} = 0$. But this form of the Gauss-Bonnet formula does allow us to observe that: *If* $\dim E_{\lambda_{1,0}}$ *is even (this is automatic) and positive then there exists a point* $p \in M$ *such that* $K(p) \leq 0$.

In case $\dim E_{\lambda_{1,0}} > 0$, M is not a sphere. So these results tell us how to get some non-positive curvature by adding handles to the sphere.

If we don't want to add handles to the sphere, it is Corollary 5.6 which tells us, at least in the case of surfaces of revolution, how to obtain some negative curvature by changing the dimension of the euclidian space into which it embeds.

Collecting the forgoing ideas together, one can state a result which gives a unified, if not quite complete, answer to the question of the existence of non-positive curvature, in other words: a generalization of Proposition 6.1 which includes surfaces with Euler characteristic 2.

Corollary 6.2. *Let* (M, g) *be an orientable, compact, boundaryless surface with metric* g*, isometry group* $\Im(M, g)$ *and* j*-th distinct* q*-form eigenvalue* $\lambda_{q,j}$*. If, for some* $q \in \{|\dim \Im(M, g) - 1|, 1\}$*,* $\dim E_{\lambda_{q,|1-q|\cdot j}}$ *is even and positive for all* j *such that* $1 \leq j \leq 4$*, then there exists a point* $p \in M$ *such that* $K(p) \leq 0$.

Proof. If (M, g) satisfies the hypothesis for $q = 1$ then the statement of this result is simply Proposition 6.1 as can be seen from Equation (6.1). Hence there exists a point $p \in M$ such that $K(p) \leq 0$.

If (M, g) satisfies the hypothesis for $q = |\dim \Im(M, g) - 1|$, then as is well known, we must consider all cases with $\dim \Im(M, g) \leq 3$. If $\dim \Im(M, g) = 0$ or 2, then, again, $q = 1$ and the proof is the same as the previous case. If $\dim \Im(M, g) = 3$ then $(M, g) = (S^2, can)$ (see [13], p. 46, 47) so that $K > 0$ and constant, and the statement of this result is simply, as we already know, that one of the first four, 0 or 2-form, eigenvalues has odd multiplicity (in fact they all have odd multiplicity). Finally, if $\dim \Im(M, g) = 1$ then $q = 0$. If, in this case, the hypothesis holds for $q = 0$ only then M is, topologically, the sphere and thus the statement of this theorem reduces to Corollary 5.6. □

One cannot help but ponder the possibility that one can remove the, a priori, assumption of S^1 invariance since, according to legend, only surfaces of revolution would have a lot of even multiplicities anyway. Also, if we might hazard an even more provacative conjecture: perhaps there is a formula which relates integrals of geometric invariants with multiplicities of non-zero eigenvalues in a similar way as Formula (6.1). One could perhaps use heat kernel asymptotics to explore this. But this would be the subject of another treatise.

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